Parking Functions: An Exploration into Peaks

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Abstract

In 2013, Billey and Burdzy [1] proved that there are 2^{n-1} permutations of length n with no peaks. We discuss generalizations of their results where instead of permutations, we look at parking functions. Since parking functions can have repeating digits, we discuss approaches to enumerate parking functions with no peaks. In particular, we look at specialized cases and their structures with respect to peaks, valleys and plateaus.

Background

Consider n parking spaces with n cars on a one-way street, where the i-th car is denoted c_i. A **parking preference** α is an ordered n-tuple where the i-th component corresponds to the preferred parking spot of c_i. A parking preference is a **parking function** of length n if and only if its non-decreasing arrangement $\beta = (b_1, \ldots, b_n)$ has the property that $b_i \leq i$ for all $i \in [n]$.

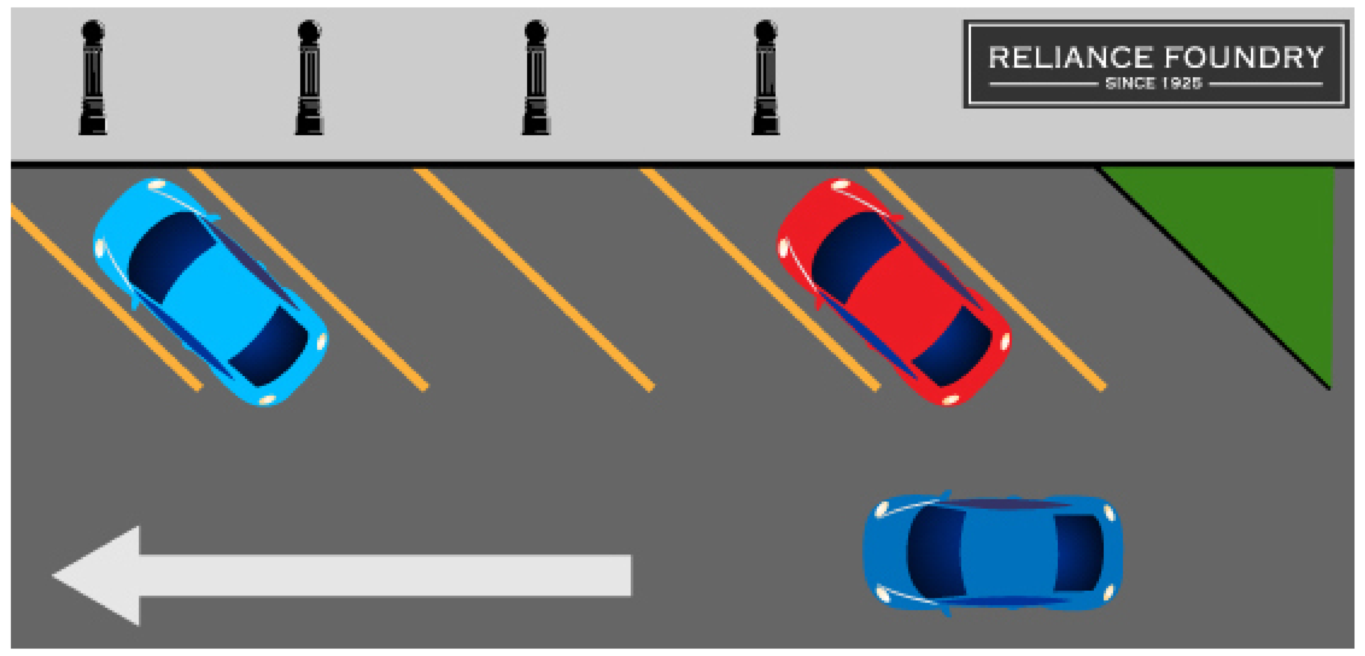


Figure 1: Car driving down a one-way street. Image from https://tinyurl.com/yxnp5yy2

If $\alpha = (a_1, a_2, \cdots, a_n)$ is a parking preference, then a **step** of α is a pair of adjacent entries (a_i, a_{i+1}) of α. There are three types of steps, illustrated below.

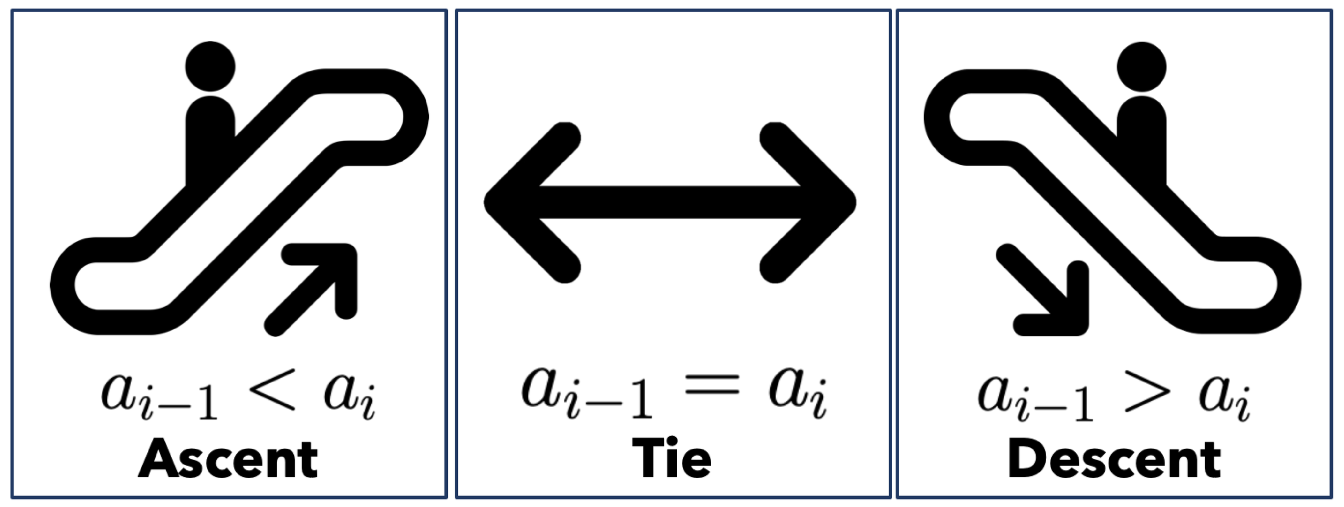
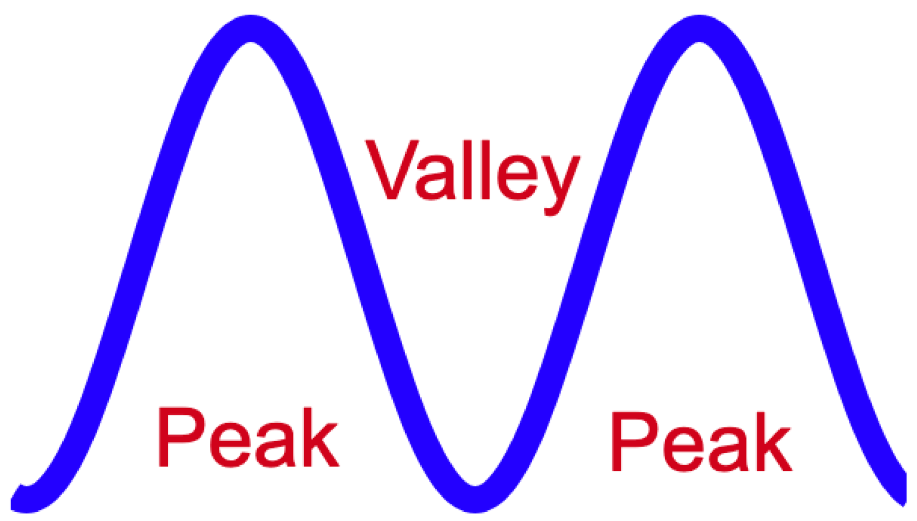


Figure 2: Three types of steps. Image from stickpng.com

A **peak** exists in a parking preference when $a_i > a_{i-1}, a_{i+1}$, i.e. when there is an ascent directly followed by a descent. Similarly, a **valley** exists when $a_i < a_{i-1}, a_{i+1}$.



Results

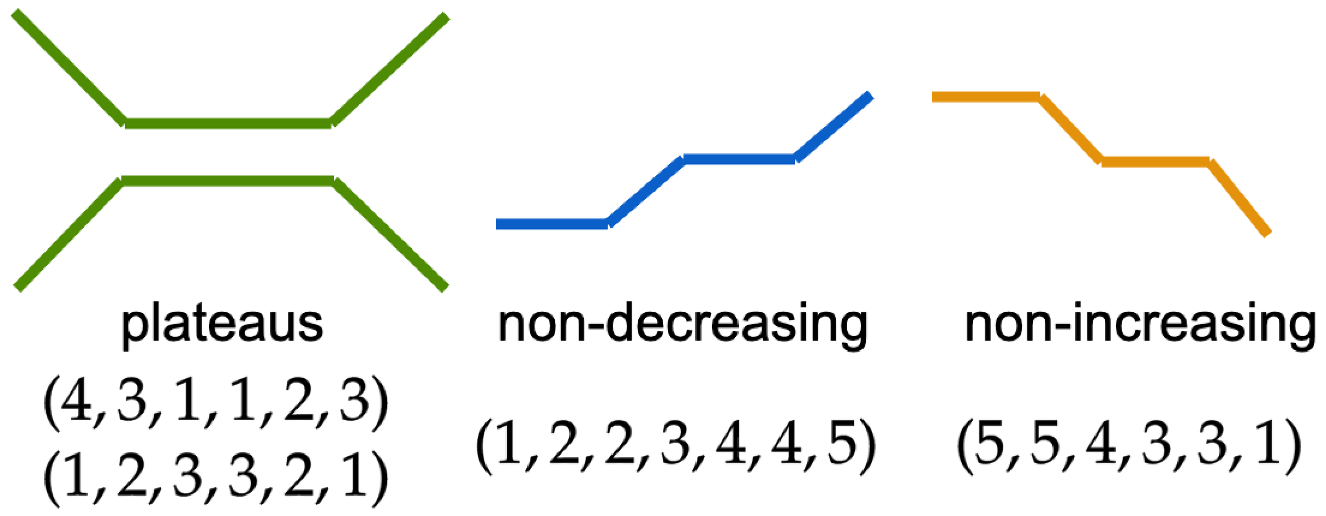
Permutations vs Parking Functions:

Permutations are rearrangements of $\{1, 2, 3, \cdots, n\}$. This means there are no repeating numbers! However, in a parking function, repeating numbers are possible.

The BIG Research Question:
How many parking functions of length n have **no** peaks?

Shapes and Structures:

There are certain patterns of consecutive ascents, descents and ties that will result in a parking function with no peaks. These look like...



Special Cases:

1. How many rearrangements of $(1, 1, 2, 3, \cdots, n)$ have no peaks?
2. How many rearrangements of $(\underbrace{1, \cdots, 1}_{k}, 2, 3, \cdots, n)$ have no peaks?

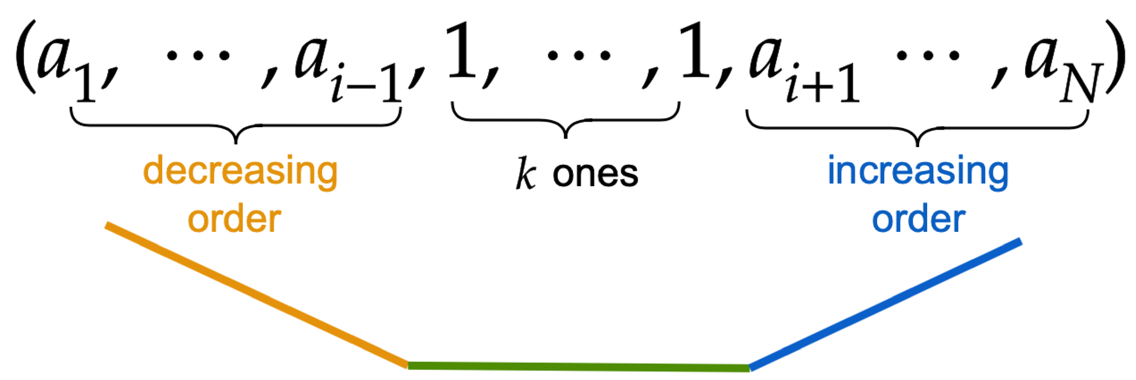
Theorem 1. *Let $\alpha = (\underbrace{1, \cdots, 1}_{k}, 2, 3, \cdots, n)$ be the non-decreasing parking function of length $N = n + k - 1$ consisting of k 1's followed by the numbers $2, 3, \ldots, n$. Then there are*

$$\sum_{i=1}^{n-1} \binom{n-2}{i-1} = 2^{n-2}.$$

rearrangements of α with no peaks. Substituting in $N = n + k + 1$ we get 2^{N-k}.

Sketch of Proof: We consider the situations where the arrangements have no peaks.

- The 1's can never be separated by other numbers; let the first placed 1 be at index i.
- There are $\binom{n-1}{i-1}$ ways to select numbers to fill the spots at indices 1 through $i-1$ inclusive, but only one way to order them.
- There is one way to place the remaining numbers in increasing order at indices $i+1$ through N inclusive.



The Next Question: What happens if instead of having repeating 1's, we have a different number repeat? What other shapes do we get and how do we count them?

Conclusion & Open Problems

Throughout our project, we looked at these various patterns and found some new sequences.

The Sequence	Elements $n \geq 3$
$\lvert P_*(\emptyset; n)\rvert$	12, 59, 351, 2499
$\lvert P_R(\emptyset; n)\rvert$	8, 51, 335, 2467, 20759
$\lceil K - \#P_*(\emptyset; n)\rceil$	4, 4, 297, 5905, 11024
$\lvert P_R(\{2\}; n+1)\rvert - \lvert P_R(\{2\}; n)\rvert$	23, 186, 1486, 13137
$\lvert P_R\{\emptyset; n\}\rvert - n^{(n-1)}$	1 , 13, 290, 5309
$\lvert P_R\{\emptyset; n\}\rvert - n!$	2, 27, 215, 1747

We found new conditions that parking functions must satisfy to contain no peaks and considered the cases where specific numbers repeat in a parking function. Look for our submission of these new sequences to OEIS. Also, you can find code for enumerating parking functions with no peaks, parking functions with peaks at a given index i, and more on GitHub.

Open Problems:

1. We've looked at what happens when only one number repeats. What about k numbers?
2. A generalization of the set of parking functions, known as the *Naples parking functions*, allows a car to park in the spot directly behind its preferred spot before moving forward if their preferred parking spot is taken. How many Naples parking functions have no peaks?
3. How many parking functions of length n have no valleys?

References

[1] Sara Billey, Krzysztof Burdzy, and Bruce E. Sagan, *Permutations with given peak set*, J. Integer Seq. **16** (2013), no. 6, Article 13.6.1, 18. MR3083179.

[2] Weisstein, Eric W. "Binomial Sums." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/BinomialSums.html

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