PROPERTIES OF PEAKS IN PARKING FUNCTIONS

MAYLEEN CORTEZ, SUMMAR ELLIS, PAMELA E. HARRIS, MORGAN HOBSON, MARISSA LOVING, JOSEPH RENNIE, AND GORDON ROJAS KIRBY

ABSTRACT. In 2013, Billey, Burdzy, and Sagan proved that there are 2^{n-1} permutations of length n with no peaks. In this paper, we discuss generalizations of their results where instead of permutations, we investigate parking functions with no peaks. In particular, we study certain subsets of parking functions and enumerate peaks by analyzing their valleys and plateaus. We also analyze the bijection between nondecreasing parking functions with repeating digits and certain labeled Dyck paths.

1. Introduction

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ denote the set of natural numbers and for $n \in \mathbb{N}$, let $[n] = \{1, \ldots, n\}$. We let S_n denote the set of permutations of length n, where we use one-line notation within an n-tuple to represent each permutation. We call an index i a **peak** in a permutation $(a_1, \ldots, a_n)$ if $a_{i-1} < a_i > a_{i+1}$. In 2012, Billey, Burdzy, and Sagan established that there are 2^{n-1} permutations with no peaks. This paper generalizes the work of Billey et al. by investigating peaks in parking functions [2]. Parking functions were first introduced by Konheim and Weiss in 1966 while studying a computer science problem known as the hashing problem [4]. In their paper, they showed that the number of parking functions of length n is given by $(n+1)^{n-1}$.

In our paper, we first introduce properties of parking functions, properties of peaks in permutations and properties of peaks in parking functions (Section 2). Next, we introduce the motivating research question for our paper:

Question 1.1. How many parking functions have no peaks?

In Section 3, we use computational tools to examine the number of parking functions with no peaks. This led to discovering new integer sequences not currently listed in the OEIS. One further area of research would be to find bijections between these parking functions with no peaks and other known families of combinatorial objects. In Sections 4 and 5, we focus our study on the subset of parking functions that have repeating entries. In particular we give results when the parking functions have a fixed number of repeated ones and the remaining digits are all distinct. In this specialization, we give formulas for counting the number of such parking functions with no peaks. In Section 6, we analyze labeled Dyck paths to illustrate the work in the previous sections involving parking functions with repeated ones.

By understanding the behavior of parking functions with repeated ones and no peaks, we aim to generalize these results to other types of parking functions with repeating digits. This get us closer to answering Question 1.1.

2. Background

For $n \in \mathbb{N}$, we consider n parking spaces and n cars on a one-way street, where the i-th car is denoted by c_i. A **parking preference** α is an ordered n-tuple where the i-th entry corresponds to the preferred parking spot of c_i.

Let $PP_n = \{(a_1, \ldots, a_n) : a_i \in [n] \text{ for all } i\}$ be the set of all parking preferences of length n. To define parking functions we first establish the following rule (illustrated in Figure 1): car c_i first drive east to its preferred parking spot k. If the k^{th} parking spot is already taken, then it proceed to the east until it reaches the next available parking spot where it then park. If every parking spot from the k^{th} spot through the n^{th} is taken, then car c_i is unable to park since each car can only ever move forward, as we are on a one-way street. We say that $\alpha \in PP_n$ is a **parking function** if each car c_i is able to park either in its preferred parking spot or in a parking spot to the east of its preferred spot.

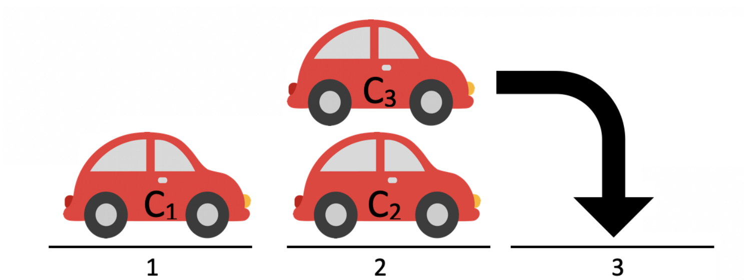


FIGURE 1. An illustration depicting the rules of parking functions for parking preference $(1, 2, 2)$.

Let PF_n be the set of all parking functions of length n. The following is a useful characterization of the parking preferences which yield parking functions.

Theorem 2.1. A parking preference α is a parking function of length n if and only if its non-decreasing arrangement $\beta = (b_1, \ldots, b_n)$ has the property that $b_i \leq i$ for all $i \in [n]$.

Proof. Let $\alpha = (a_1\ a_2 \ldots a_3)$ be a parking preference of length n and β be the increasing rearrangement of α such that $b_1 \leq b_2 \leq \cdots \leq b_n$.

First assume that $\alpha \in PF_n$. We want to show that for any $i \in \mathbb{N}$ we have $b_i \leq i$. Assume, towards contradiction, that $\alpha \in PF_n$ and that there is an index i such that $b_i > i$. Let $k \in \mathbb{N}$ be the smallest index where $b_i > i$. In other words, we have that $b_k > k$ and for any index $j \geq k, j \in \mathbb{N}$ we have $b_j \geq k$. So we have $n - k$ cars but only $n - k - 1$ parking spots. So, at least one car does not have a parking spot, a contradiction.

Now assume that for all $i \in \mathbb{N}$ we have $b_i \leq i$. We want to show that $\alpha \in PF_n$. To that end, let i be an arbitrary index. Consider where the i^{th} car park. Since there are $i - 1$ cars that came before it and since $b_j \leq j\ \forall j$, we have that parking spots 1 through $i - 1$ are filled. If $b_i < i$, then car i will move to the left until it reaches the i^{th} spot and parks there. Otherwise, $b_i = i$ and since the i^{th} is empty, the i^{th} car will park there. Thus, since i was arbitrary, we have that for any index i, car i will park in the i^{th} spot. Thus, α is a parking function. ☐

One example of a parking function is the parking preference $\alpha = (a_1, \ldots, a_n)$ such that $a_i = 1$ for any $i \in [n]$. Then the increasing rearrangement of α is equal to α, and $a_i = 1 \leq i$ for any index i. Thus, $\alpha \in PF_n$.

We establish that given a parking function $(a_1, a_2, \ldots, a_k, \ldots a_n)$, if we replace a_k with some value $a'_k \in [n]$ with $a'_k < a_k$, then $(a_1, a_2, \ldots, a'_k, \ldots, a_n)$ is still a parking function. Before presenting a proof we illustrate this with some examples.

Example 2.2. Let $\alpha = (1, 2, 3, 4, 5)$. Note that α is in increasing order, so for each index i, $a_i \leq i$. Hence, α is a parking function by Theorem 2.1. We create some new parking preferences by modifying α. Consider the following parking functions

$$\beta = (1, 2, 2, 4, 5) \text{ and } \gamma = (1, 2, 3, 4, 3).$$

Notice that α and β are identical except at index $i = 3$, where we have $b_3 = 2 < 3 = a_3$. Similarly, α and γ are identical except at index $i = 5$ where we have $c_5 = 3 < 5 = a_5$. Note that β is in increasing order, and $b_i \leq i$. Hence, β is a parking function of length 5. Now, let $\gamma' = (1, 2, 3, 3, 4)$ be the increasing arrangement of γ. Using a similar argument as before, we see that γ is also a parking function of length 5.

Lemma 2.3. Let $\alpha = (a_1, a_2, \ldots, a_k, \ldots, a_n) \in PF_n$. If we replace a_k with some value $a'_k \in [n]$ with $a'_k < a_k$, then $(a_1, a_2, \ldots, a'_k, \ldots, a_n) \in PF_n$.

Proof. Consider an arbitrary parking function $\alpha = (a_1, a_2, \ldots, a_k, \ldots, a_n) \in PF_n$ and let $\beta = (b_1, \ldots, b_n)$ be its non-decreasing arrangement. Let $\alpha' = (a'_1, a'_2, \ldots, a'_k, \ldots, a'_n)$ such that $a'_k < a_k$ and, for any $j \in [n] \setminus \{k\}$, we have $a'_j = a_j$. Hence, we have $a'_i \leq a_i$ for all $i \in [n]$. Now, let $\beta' = (b'_1, \ldots, b'_n)$ be the non-decreasing arrangement of α'. Then, we have $b'_i \leq b_i \leq i$ for all $i \in [n]$. Thus, for any $i \in [n]$, we have $b'_i \leq i$. Therefore, by Theorem 2.1, α' is a parking function of length n. ☐

We also observe that only one element in a parking function of length n can equal n. In other words, if n appears in a parking function at all, it only appears once. We include a proof and some examples below.

Lemma 2.4. Consider $\alpha = (a_1, a_2, \ldots, a_n) \in PF_n$. If there exists some $j \in [n]$ such that $a_j = n$, then for all indices $i \neq j$, we have $a_i < n$.

Proof. Consider an arbitrary parking function $\alpha = (a_1, a_2, \ldots, a_n)$ of length $n \in \mathbb{N}$. A car's preference $a_i = n$ can appear only once in a parking function. If not, we have some parking function $\alpha = (a_1, a_2, \ldots, a_n)$ such that there exist indices j and k where $a_j = n = a_k$. Then its non-decreasing rearrangement $\beta = (b_1, \ldots, b_{n-1}, b_n)$, where $b_{n-1} = n = b_n$ and $n - 1 < b_{n-1}$ and $b_n = n$. This contradicts the assumption that α is a parking function where $i \geq b_i$. Hence, n can only appear once in a parking function. ☐

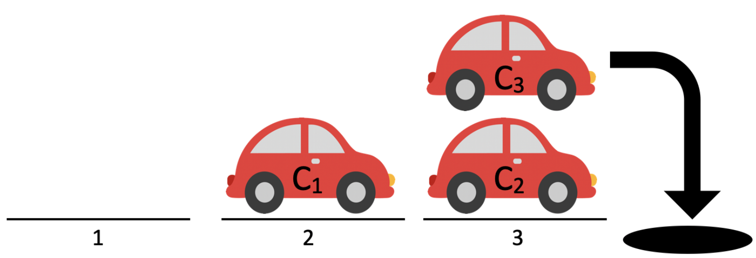


FIGURE 2. An illustration depicting the rules of parking functions for parking preference $(2, 3, 3)$.

The following is an example to demonstrate why n can only appear once in a parking function.

Example 2.5. Consider the parking preference $(2, 3, 3)$ of length $n = 3$. Car one prefers spot 2. Since it is empty, it parks there. Car two prefers spot 3. Since it is empty, it parks there. Car three prefers spot 3. However, since car two is parked there, it proceed east on the one way street to the first available spot. However, there are no spots beyond parking spot n, so car three is unable to park.

Definition 2.6. A **permutation** is a rearrangement of $\{1, 2, \ldots, n\}$ where no elements repeat and n is a natural number. The set of all permutations of a given length n is denoted as S_n or $\mathfrak{S}_n$. There are $n!$ permutations of $[n]$.

Notice that any permutation of length n is also a parking function of length n. Since every element in a permutation is distinct, every car will be able to park in their preferred parking spot. Hence, we have $S_n \subset PF_n$.

Example 2.7. Consider S_3:

$$S_3 = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}.$$

Observe that when the entries of a permutation are arranged in increasing order, each permutation becomes $(1,2,3)$. This follows the condition of the i-th element being less than or equal to i. Hence, all elements of S_3 are parking functions.

Definition 2.8. If $\alpha = (a_1, a_2, \ldots, a_n)$ is a parking preference, then a **step** of α is a pair of adjacent entries (a_i, a_{i+1}) of α. There are three types of steps. When $a_i < a_{i+1}$, we have an **ascent**. When $a_i = a_{i+1}$, we have a **tie**. A **plateau** is a sequence of ties. When $a_i > a_{i+1}$, we have a **descent**. We say there is a **peak** at index i when $a_i > a_{i-1}, a_{i+1}$. We say there is a **valley** at index i when $a_i < a_{i-1}, a_{i+1}$.

Example 2.9. Consider the parking preference $\alpha = (1,4,4,3)$ of length $n = 4$. There are $n-1 = 4-1 = 3$ steps. The first step, $(1,4)$, is an ascent. The second step, $(4,4)$, is a tie. The third step $(4,3)$ is a descent.

We introduce notation for counting peaks in permutations. For a permutation of length n, let $I \subseteq \{2,3,\ldots,n-1\}$ denote the set of indices where peaks occur. Notice we never have $1 \in I$ nor $n \in I$ since a peak cannot occur at index 1 or n, which we establish in Lemma 2.11. We use $P(I;n)$ to denote the set of permutations of length n with peaks exactly at indices $i \in I$. From this point forward, we use the notation # to denote the cardinality of a set. Thus, the cardinality of the set $P(I;n)$ is denoted $\#P(I;n)$.

Example 2.10. Consider the following:

$$S_3 = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}.$$

Note that there are two permutations with peaks at $i = 2$, which are highlighted in red above. Thus

$$P(\{2\};3) = \{(1,3,2),(2,3,1)\} \text{ and } \#P(\{2\};3) = 2.$$

Also,

$$P(\emptyset;3) = \{(1,2,3),(2,1,3),(3,1,2),(3,2,1)\} \text{ and } \#P(\emptyset;3) = 4.$$

Thus, there are four permutations with no peaks.

In S_5, there are 16 permutations with peaks exactly at the indices 2 and 4 (e.g. $(1,4,3,5,2)$). Therefore $\#P(\{2,4\};5) = 16$.

The following are remarks on the properties of peaks. The first to note is where peaks can never occur.

Lemma 2.11. Consider $\alpha = (a_1, a_2, \ldots, a_n) \in PF_n$. There can never be a peak at the first index nor at the n-th index.

Proof. Consider an arbitrary parking function $\alpha = (a_1, a_2, \ldots, a_n)$ of length $n \in \mathbb{N}$. Recall that by definition, there is a peak at index i if $a_i > a_{i-1}, a_{i+1}$. If we have a peak at $i = 1$, then $a_1 > a_0, a_2$. However, there is no a_0 so we cannot satisfy the condition to have a peak at i. Similarly, if $i = n$ there is no a_{n+1} to compare to and hence, there can never be a peak at index n. □

Lemma 2.12. Consider $\alpha = (a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_n) \in PF_n$. If there is a peak at index i, there cannot be a peak at indices $i-1$ or $i+1$.

Proof. Consider an arbitrary parking function $\alpha = (a_1, a_2, \ldots, a_n)$ of length $n \in \mathbb{N}$. Assume, that a_i and a_{i+1} are both peaks for some $i \in [n]$ to obtain a contradiction. By definition of a peak, $a_{i-1} < a_i > a_{i+1}$. Since a_{i+1} is also a peak, then $a_{i+1} > a_{i+2}$ and $a_{i+1} > a_i$. This contradicts the fact that a_i is a peak and $a_i > a_{i+1}$. Therefore, peaks cannot be adjacent to each other. □

We now note that given a parking function of length n, since a peak cannot be in the first or n-th indices, the only indices where peaks can occur are at $2 \leq i \leq n-1$. We are interested in examining peaks in parking functions. Hence, we look for properties of peaks in parking functions to better understand how they behave.

Lemma 2.13. Consider $\alpha = (a_1, a_2, \ldots, a_n) \in PF_n$. If there is a peak in α, then $n \geq 3$.

Proof. Consider an arbitrary parking function $\alpha = (a_1, a_2, \ldots, a_n)$ of length $n \in \mathbb{N}$. If $1 \leq n < 3$, then the only possible positions for a peak are $i = 1$ or $i = 2$, yet we know peaks cannot occur at either of these indices. Therefore, no peaks exist if $1 \leq n < 3$. The □

One final observation about peaks in parking functions is that if n is an entry in an index $i \in \{2, \ldots, n-1\}$, then there is a peak at index i. This is in addition to the Lemma 2.4 where $a_i = n$ can only occur once.

Lemma 2.14. Consider $\alpha = (a_1, a_2, \ldots, a_n) \in PF_n$. If there exists an index $i \in \{2, 3, \ldots, n-1\}$ such that $a_i = n$, then there is a peak at index i.

Proof. Consider an arbitrary parking function $\alpha = (a_1, a_2, \ldots, a_n)$ of length $n \in \mathbb{N}$. Let there be some index $i \in \{2, 3, \ldots, n-1\}$ where $a_i = n$. The greatest value a component of α can have is n. However, $a_n = n$ can only occur once in a parking function (refer to Lemma 2.4). Thus, we know that $a_{i-1}, a_{i+1} < n$. Since $a_i = n$, then $a_{i-1} < n > a_{i+1}$. Therefore, we have a peak at index i. □

Having established some conditions about peaks in parking functions, we are now interested in counting the number of parking functions with no peaks. This will be a generalization of the work done by Billey, Burdzy and Sagan in 2013 on enumerating the number of permutations of length n with zero peaks [2]. For ease of reference, we restate their result below and provide a proof.

Theorem 2.15 (Billey, Burdzy, and Sagan 2013 [2])**.** The number of permutations of length $n \geq 3$ with no peaks is 2^{n-1}.

Proof. Let $n \geq 3$. Recall that a peak is an ascent step directly followed by a descent step. Consider the permutation $\alpha = (a_1, a_2, \ldots, n) \in S_n$. If α has no peaks, then with a one in the i-th entry $\alpha = (a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n)$, where $a_1, \ldots, a_{i-1}$ are arranged in descending order and $a_{i+1}, \ldots, a_n$ are in ascending order. Note that if $i = 1$, there are no numbers preceding the 1 in the permutation. If $i = n$, there are no numbers following it in the permutation.

There exist n choices for the placement of 1. If 1 is placed at index i, $1 \leq i \leq n$, then we select $i-1$ numbers from $n-1$ available numbers to fill the $i-1$ spaces before the 1 at spot i. There are $\binom{n-1}{i-1}$ ways of doing so. The arrangement of those $i-1$ numbers preceding the 1 is determined since they must be in descending order. The arrangement of the $n-i-1$ numbers not chosen that follow the 1 is determined since they must be in ascending order. Thus, summing over the possible placements of the 1 in spot $i = 1$ to $i = n$, we get

$$\sum_{i=1}^{n} \binom{n-1}{i-1}.$$

This enumerates the permutations of length n without peaks. Using the Binomial Theorem[1], we have

$$\sum_{i=1}^{n} \binom{n-1}{i-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} = (1+1)^{n-1} = 2^{n-1}.$$

□

We are interested in determining a closed formula for the number of parking functions of length n with no peaks, denoted $\#P_*(\emptyset, n)$. There are $(n+1)^{n-1}$ total parking functions of length n and there are $n!$ total permutations of length n. Notice that $S_n \subset PF_n$ implies that $\#P(\emptyset; n) < \#P_*(\emptyset; n)$ and since there is a known formula for enumerating permutations of length n with no peaks, we choose to only consider parking functions of length n that are not permutations. We call these: "parking functions with repeating digits." The set of these is denoted RPF_n. Therefore, there are $(n+1)^{n-1} - n! = \#RPF_n$ total parking functions with repeating digits of length n. We are interested in determining a closed formula for the number of parking functions with repeating digits of length n with no peaks, denoted $\#P_R(\emptyset, n)$. Thus,

$$\#P_*(\emptyset; n) = \#P_R(\emptyset; n) + \#P(\emptyset; n) = \#P_R(\emptyset; n) + 2^{n-1}.$$

3. **Sequences**

In this section, we describe our work to generate terms of sequences related to counting peaks in parking functions. Through some computational experimentation, we created some sequences of positive integers which describe the number of parking functions with certain properties related to the number of peaks. The code for our computations can be found on GitHub[2], to calculate the number of parking functions with specific peak sets. These sequences were then compared to known sequences in the OEIS. Unfortunately, none of our sequences appeared in this encyclopedia. Thus, more work is required to determine closed formulas that enumerate these sequences. Moreover, our goal was to find other families of combinatorial objects that may be in bijection with parking functions with certain peaks. In particular, it can be used to compute the first terms of the sequence $P_*(\emptyset; n)$: $12, 59, 351, 2499, 20823, \ldots$.

Since every permutation of length n is a parking function, and the number of permutations with no peaks has been previously enumerated, we focus on parking functions in the set of parking functions with repeating digits, (i.e. that are not permutations). Table 1 compares the number of parking functions, the number of permutations, and the number of parking functions with repeating digits. We were able to compute the number of parking functions of length n with repeated digits for small values of $n \geq 3$, specifically $8, 51, 335, 2467,$ and 20759. For terms of the new sequences we found, refer to Table 9 in the Appendix.

3.1. **Peaks at index *i*.**

In this section, we make observations about peaks at specific indices in parking functions. Table 2 shows computed values for the number of parking functions with repeating digits that have no peaks or exactly one peak at some index $i \in \{2, 3, \ldots, 7\}$. Table 3 shows computed values for the number of parking functions with repeating digits, which have peaks at specified indices.

We also worked on enumerating the number of ways that exactly one, exactly two, or exactly three peaks could appear in a parking function.

Example 3.1. Consider $\alpha = (a_1, a_2, \ldots, a_5) \in PF_5$. If there is only one peak in α, that peak may occur at any index $i \in \{2, 3, 4\}$, per Lemma 2.11. Thus, we say that for PF_5, there are three **singles**. For any length n, there are $n-2$ singles. This is a direct consequence of Lemma 2.11.

[1] See http://mathworld.wolfram.com/BinomialSums.html

[2] https://github.com/mayscortez/Peaks-in-Parking-Function

n	$PF_n = (n+1)^{(n-1)}$	$\#S_n = n!$	$\#RPF = (n+1)^{n-1} - n!$
1	1	1	0
2	3	2	1
3	16	6	10
4	125	24	101
5	$1,296$	120	$1,176$
6	$16,807$	720	$16,087$
7	$262,144$	$5,040$	$257,104$
8	$4,782,969$	40320	$4,742,649$
9	$100,000,000$	$362,880$	$99,637,120$

TABLE 1. The $\#PF_n$, $\#S_n$ and $\#RPF_n$ for a given length, n

n	$P_R(\emptyset;n)$	$P_R(\{2\};n)$	$P_R(\{3\};n)$	$P_R(\{4\};n)$	$P_R(\{5\};n)$	$P_R(\{6\};n)$	$P_R(\{7\};n)$
3	8	2	—	—	—	—	—
4	51	25	25	—	—	—	—
5	335	211	315	211	—	—	—
6	2467	1697	2947	2947	1697	—	—
7	20759	14834	26673	31054	26673	14834	—
8	197437	146951	261950	314498	314498	261950	146951

TABLE 2. Peaks only at index i

n	$P_R(\{2,4\},n)$	$P_R(\{3,5\},n)$	$P_R(\{2,5\},n)$	$P_R(\{2,4,6\},n)$	$P_R(\{2,6\},n)$	$P_R(\{3,6\},n)$	$P_R(\{4,6\},n)$
5	104	—	—	—	—	—	—
6	$1,541$	$1,541$	1250	—	—	—	—
7	$16,656$	$25,854$	$21,037$	$9,198$	$11,839$	$21,037$	$16,656$
8	$171,308$	$314,907$	$258,598$	$178,341$	$223,856$	$402,197$	314907

TABLE 3. Peaks at multiple indices i

Similarly, the only way we can have two peaks in α is if they occur at indices 2 and 4 so there is one **double**. However, there is no way that we can have three peaks in a parking function of length 5 so we have no **triples**.

Example 3.2. Consider $\alpha \in PF_6$. If there are exactly two peaks in α, then those peaks may occur at the following pairs of indices: $(2,4)$, $(2,5)$, or $(3,5)$. Hence, we say that for PF_6, there are three doubles. We conjecture that doubles can be counted by the triangular numbers (OEIS A000217).

Example 3.3. Consider $\alpha \in PF_7$. If there are exactly three peaks in α, then those peaks may occur at the indices 2, 4 and 6. Hence, we say that for PF_6, there is one triple.

Table 4 summarizes this data that for parking functions of length $4 \leq n \leq 9$. Note that in the row corresponding to $n = 9$ the sum of the number of singles, doubles and triples is 32. However, there is one unique choice of indices where four peaks can occur, indices $\{2,4,6,8\}$. Thus, the Total column for $n = 9$ is 33.

3.2. **The Maximum Number of Peaks in a Parking Function.** Since a peak cannot occur in the first or n-th indices, there are $n-2$ places where a peak can possibly occur (Refer to Lemma

n	Single	Double	Triple	Total
4	2	0	0	2
5	3	1	0	4
6	4	3	0	7
7	5	6	1	12
8	6	10	2	18
9	7	15	10	33

TABLE 4. This table gives number of ways that exactly one, exactly two, or exactly three peaks could appear in a parking function.

2.11). However, two peaks cannot be adjacent to each other. Thus, the maximum number of peaks is less than $n-2$ (Refer to Lemma 2.12).

Example 3.4. Consider the following parking functions of length 5 and where they have peaks:

$\alpha = (2, 4, 5, 3, 1)$ has one peak at index $i = 3$
$\alpha' = (2, 4, 3, 5, 1)$ has two peaks at indices $i = 2$ and $i = 4$

The maximum number of peaks for a parking function of length 5 is two. Now consider where multiple peaks can occur in PF_6:

$\beta = (1, 4, 2, 6, 5, 3)$ has peaks at indices $i = 2$ and $i = 3$
$\beta' = (1, 2, 4, 5, 6, 3)$ has peaks at indices $i = 3$ and $i = 5$
$\beta'' = (1, 4, 2, 3, 6, 5)$ has peaks at indices $i - 2$ and $i = 5$

The maximum number of peaks for a parking function of length $n = 7$ is three. Now consider where multiple peaks can occur in parking functions of length 7 with distinct values:

$\gamma = (1, 2, 4, 3, 6, 5, 7)$ has two peaks at indices $i = 3$ and $i = 5$
$\gamma = (2, 5, 4, 7, 3, 6, 1)$ has 3 peaks at indices $i = 2$, $i = 4$, and $i = 6$

The maximum number of peaks for a parking function of length 7 is three.

In the case of parking functions of length 6, there are multiple combinations of indices that, if peaks occur at them, the parking function have the max number of peaks possible. This is different than when $n = 5$ and $n = 7$ where there is only one combination of peak occurrences to get the maximum. Thus we observed that when n is odd, the max number of peaks occur at the even indices. From this, we began to look at PF_n when n is odd.

Table 5 shows the pattern for the maximum number of peaks when n is odd, i.e. where $n = 2k + 1$ such that $k \in \mathbb{N}$. Notice that k (second column) is equal to the maximum peaks (third column).

$n = 2k + 1$	k	Max Peaks
1	0	0
3	1	1
5	2	2
7	3	3
9	4	4
11	5	5
13	6	6
15	7	7

TABLE 5. The max number of peaks when $n = 2k + 1$

After solving for k in terms of n, we find that $k = \frac{n-1}{2}$. Hence, we have a closed formula for the maximum number of peaks in a parking function of odd length n. If we take a look at the patterns for even n, we see that the maximum number of peaks for a parking function of length n is

$$\left\lfloor \frac{n-1}{2} \right\rfloor.$$

We include some examples and then a proof. This closed form holds for the parking functions α, γ and β from Example 3.4.

$$\begin{aligned} \text{For } \alpha: &\ \lfloor \tfrac{5-1}{2} \rfloor = 2 \\ \text{For } \gamma: &\ \lfloor \tfrac{7-1}{2} \rfloor = 3 \\ \text{For } \beta: &\ \lfloor \tfrac{6-1}{2} \rfloor = 2 \end{aligned}$$

Afterwards, we saw that to enumerate the maximum possible number of peaks that can occur in parking functions of length n, we can use the formula

$$\left\lfloor \frac{n-1}{2} \right\rfloor (n+1)^{n-1}.$$

This formula counts the total possible number of peaks in every parking function of length n. It is also an upper bound for the number of peaks that occur in PF_n. However, it is a larger upper bound than we needed since,

$$\#(PF_n \setminus P_*(\emptyset; n)) \leq (n+1)^{n-1} \leq \left\lfloor \frac{n-1}{2} \right\rfloor (n+1)^{n-1}.$$

It is not an upper bound for the number of parking functions of length n with *any* peaks. By *any* peaks, we mean parking functions with at least one peak regardless of index, denoted as $PF_n \setminus PF_n(\emptyset)$. The cardinality of $PF_n \setminus PF_n(\emptyset)$ must be less than or equal to $\#PF_n$.

Lemma 3.5. Consider the parking funtion $\alpha = (a_1, a_2, \ldots, a_n)$. The maximum number of peaks in α is

$$\left\lfloor \frac{n-1}{2} \right\rfloor.$$

The maximum number of peaks occur at non-consecutive indices $\{2, \ldots, n-1\}$. This is the same as counting the number of non-consecutive numbers $\{1, \ldots, n\}$. We provide a proof of this result below, which appeared as Lemma 3.1.2 in [3].

Lemma 3.6. The maximum number of non-consecutive integers that can be selected from the set $\{1, \ldots, n-2\}$ is

$$\left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof. Notice if n is odd, then we can choose at most $\frac{n-1}{2}$ many nonconsecutive integers from the set $\{2, \ldots, n-1\}$, in particular even indices. If n is even, then $n-1$ is odd and we can choose at most $\frac{n-2}{2}$ many consecutive indices from the set $\{2, \ldots, n-1\}$, either all the even or all the odd indices. Observe that when n is odd, $\frac{n-1}{2} = \lfloor \frac{n-1}{2} \rfloor$ and when n is even, $\frac{n-2}{2} = \lfloor \frac{n-1}{2} \rfloor$. Thus, the maximum number of peaks, $\lfloor \frac{n-1}{2} \rfloor$, occur at indices in nonconsecutive order. □

4. Non-decreasing Parking Functions

In this section, we define non-decreasing, show an example and introduce trends with respect to peaks in parking functions with repeating digits.

Definition 4.1. A parking function $\alpha = (a_1, a_2, \ldots, a_n)$ is **non-decreasing** when $a_i \leq a_{i+1}$ for all $i \in [n]$.

There are no peaks in non-decreasing parking functions because there never exist an index i such that $a_i > a_{i+1}$. By definition, the steps in a non-decreasing parking function are either ties or ascents. A similar argument can be made for valleys. Note this implies any parking function can be rearranged in a way such that no peaks occur.

Example 4.2. The rearrangements of some non-decreasing parking functions of length 4 are shown below.

$(1,1,1,2) \to$
(1,1,2,1)
(1,2,1,1)
(2,1,1,1)

$(1,2,2,2) \to$
(2,1,2,2)
(2,2,1,2)
(2,2,2,1)

$(1,2,2,4) \to$
(1,2,4,2) (4,2,1,2) (2,4,2,1) (2,4,1,2)
(1,4,2,2) (4,2,2,1) (2,1,4,2) (4,1,2,2)
(2,1,2,4) (2,2,1,4) (2,2,4,1)

$(1,1,3,3) \to$
(1,3,1,3)
(3,1,3,1)
(3,3,1,1)
(3,1,1,3)
(1,3,3,1)

$(1,1,2,2) \to$
(1,2,1,2)
(2,2,1,1)
(2,1,1,1)
(2,1,1,2)
(1,2,2,1)

By looking at the rearrangements of non-decreasing parking functions of length 4, we explore ways to count the number of parking functions with no peaks. We also look at patterns in relation to how many times a digit is repeated within a parking function (see Table 6). Based on these results, we decide to look at some special cases of parking functions with repeating digits. In particular, we look at parking functions with two ones and then generalize to parking functions with k repeating ones, where $k \in \mathbb{N}$.

5. Enumerating Special Cases of Parking Functions

In this section, we break down the original research question into specialized cases that focus on the shapes and structures of parking functions with repeating digits. This entails plotting a parking function of length n as a path with vertices at (i, a_i) for each $i \in [n]$ and an edge from (i, a_i) to

Non-decreasing PF_4	Number of Rearranged PF_n	w/o Peaks	w/ peaks	Trends
(1, 1, 1, 1)	1	1	0	One digit repeated 4x
(1, 2, 2, 2)	4	4	0	One digit repeated 3x
(1, 1, 1, 2)	4	2	2	One digit repeated 3x
(1, 1, 1, 3)	4	2	2	One digit repeated 3x
(1, 1, 1, 4)	4	2	2	One digit repeated 3x
(1, 1, 2, 2)	6	4	2	Two digits repeated 2x
(1, 1, 3, 3)	6	4	2	Two digits repeated 2x
(1, 2, 2, 4)	12	6	6	One digit repeated 2x
(1, 2, 2, 3)	12	6	6	One digit repeated 2x
(1, 1, 2, 3)	12	4	8	One digit repeated 2x
(1, 1, 2, 4)	12	4	8	One digit repeated 2x
(1, 1, 3, 4)	12	4	8	One digit repeated 2x
(1, 2, 3, 3)	12	8	4	One digit repeated 2x

TABLE 6. The above table shows the number of $\alpha \in PF_4$ that do or do not have peaks when their non-decreasing parking function is rearranged. This table also includes observations on the number of digits that are repeated.

$(i+1, a_{i+1})$ for each $i \in [n-1]$, in $\mathbb{R}^2$, to give a visual representation of patterns of ascents, ties and descents in parking functions.

Example 5.1. For the parking function $(1,2,2) \in PF_3$, we have a path from $(1,1)$ to $(2,2)$ to $(3,2)$ illustrated in Figure 3.

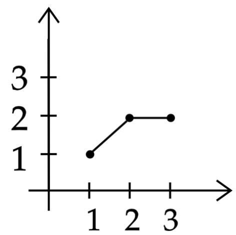


FIGURE 3. A graphic depiction of the shape of parking function (1,2,2).

Later, we look at cases where there are exactly k ones in a parking function. By focusing on special cases, we gain a better understanding of where peaks occur. This is with the intention of generalizing our findings to enumerate $\#P_R(\emptyset; n)$.

5.1. **Shapes and Structures.** There are certain patterns of consecutive ascents, descents and ties that result in a parking function with no peaks. Some of these shapes are shown in Figure 4. Underneath each general shape are examples of each respective structure. Notice that there may be combinations of these shapes that also result in parking functions with no peaks.

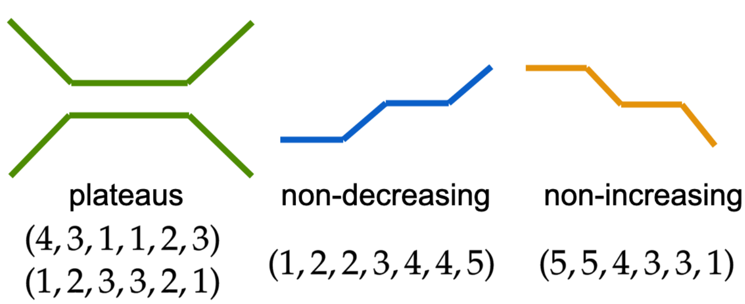


FIGURE 4. Shapes to demonstrate the structures of parking functions with repeating digits that have no peaks.

We use these ideas in the following subsection to count the parking functions described in Theorem 5.4, Theorem 5.2 and Theorem 5.7. In particular, we argue that when a parking function has two ones and the remaining elements are distinct, there is one specific pattern of ascents, ties and descent that result in no peaks. The pattern is as follows (refer to Figure 5): The ones must be in consecutive entries. Any numbers to the left of the ones should be arranged in decreasing order. The remaining numbers should be arranged in increasing order to the right of the ones.

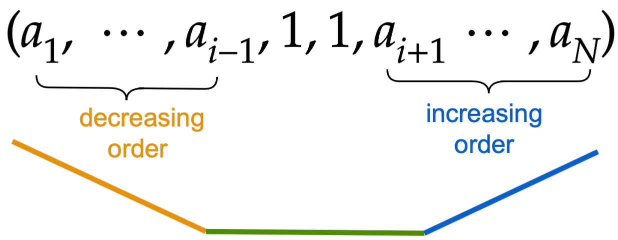


FIGURE 5. A shape to demonstrate the structure of the special case of two repeating ones.

5.2. **Repeating ones.** Here we consider parking functions where the only repeated number is the number 1. All other elements in these parking functions are distinct.

Lemma 5.2. Let $n, m \in \mathbb{N}$. Consider a parking function $\beta \in PF_N$ with n distinct entries, m entries equal to 1 and where $N = n + m - 1$. In other words, we have that β is a rearrangement of $(\underbrace{1, \dots, 1}_{k}, 2, 3, \dots, n)$. If β has no peaks, then the m entries equal to 1 are adjacent.

Proof. Let α be an arbitrary parking function of length $n+m-1$ such that $m, n \in \mathbb{N}$ and there are m elements equal to 1 but n distinct entries. Suppose there are some indices $1 \leq i, j \leq n - m - 1$ such that $i + 1 < j$, $a_i = a_j = 1$ and for all $i < k < j$ we have $1 < a_k$. In other words, we have $\alpha = (a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_{j-1}, 1, a_{j+1}, \dots, a_n)$. If $i = 1$, then there are no values to the left. If $j = n$ there are no values to the right. Since $a_{i+1}, \dots, a_{j-1} \in [n] \setminus \{1\}$ are distinct by assumption, then there exists $i + 1 \leq k \leq j - 1$ such that $a_k > a_m$ for all $i + 1 \leq m \leq j - 1$ and $m \neq k$. If $k = i + 1$, then $1 < a_k > a_{k+1}$ so a_k there is a peak at index k. If $k = j - 1$, then $a_{k-1} < a_k > 1$ so there is a peak at index k. If $i + 1 < k < j - 1$, then $a_{k-1} < a_k > a_{k+1}$ so there is a peak at index k. Thus if α has no peaks, then the k entries equal to 1 are adjacent. $\square$

Example 5.3. An example of a parking function of length 5 with four distinct entries and two entries equal to 1 is $(1, 1, 2, 3, 4)$. The following rearrangements have ones separated by other numbers:

$$(1, 2, 1, 3, 4) \Rightarrow \text{peak at } i = 2$$
$$(1, 2, 3, 1, 4) \Rightarrow \text{peak at } i = 3$$
$$(1, 2, 3, 4, 1) \Rightarrow \text{peak at } i = 4$$

Notice that in all cases, separating the ones results in a peak.

Now we look at the specific case where a parking function has two adjacent ones and all other distinct consecutive elements $\{2, 3, \dots, n - 1\}$. We want to know how many rearrangements of $\alpha = (1, 1, 2, 3, \dots, n - 1) \in PF_n$ yield no peaks.

Theorem 5.4. The number of rearrangements of $(1, 1, 2, 3, 4, \dots, n)$ with zero peaks is

$$\sum_{i=1}^{n-1} \binom{n-2}{i-1} = 2^{n-2}.$$

Proof. Let $\alpha \in PF_n$ such that $\alpha = (1, 1, a_3, a_4, \dots, a_n)$ with $a_j \in \{2, 3, ...n - 1\}$ and $a_j \neq a_k$ for $1 \leq j < k \leq n$. Consider the rearrangements of α such that the ones are always adjacent and the first 1 is located at index i $1 \leq i \leq n - 1$ (refer to Lemma 5.2).

Let i be the entry of the first placed 1 such that $\alpha = (a_1, \dots, a_{i-1}, 1, 1, a_{i+2}, \dots, a_n)$. Then, $a_1, \dots, a_{i-1}$ are arranged in descending order and $a_{i+2}, \dots, a_n$ are in ascending order. Note, if

$i = 1$, then there are no numbers preceding the adjacent ones in α and if $i = n-1$ there are no numbers following the adjacent ones in α.

There exist $n-1$ choices for the placement of the adjacent ones. If the first 1 is placed at index i, then we select $i-1$ numbers to place in the indices 1 through $i-1$. We can pick these numbers from the $n-2$ unused numbers to fill those indices. There are $\binom{n-2}{i-1}$ ways of doing so. The arrangement of those $i-1$ numbers preceding the adjacent ones is determined since they must be in decreasing order. The arrangement of the remaining $n-i-1$ numbers following the adjacent ones is so also determined since they must be in ascending order. Thus, summing over the possible placements of the adjacent ones in spot $i = 1$ to $i = n-1$ gives

$$\sum_{i=1}^{n-1} \binom{n-2}{i-1}.$$

Using the Binomial Theorem, we have that

$$\sum_{i=1}^{n-1} \binom{n-2}{i-1} = \sum_{i=0}^{n-1} \binom{n-2}{i} = (1+1)^{n-2} = 2^{n-2}$$

gives the total number of parking functions of length n with two repeating ones, distinct numbers between 2 and $n-1$ and no peaks. $\square$

Notice that in our scenario, we do not consider *all* parking functions with exactly two ones. We only consider those with consecutive numbers from 1 to $n-1$. However, what happens if we can use numbers from 1 to n? We now generalize our results to include rearrangements of $(1, 1, a_3, a_4, \ldots, a_n)$ where the elements $a_3, \ldots, a_n$ are distinct, but not necessarily consecutive (refer to Table 7).

Lemma 5.5. Consider $\alpha = (1, 1, a_3, a_4, \ldots, a_n) \in PF_n$ where for all indices $i \geq 3$, we have $a_i \in \{2, 3, \ldots, n\}$ and there are no indices j, k such that $a_j = a_k$. The number of rearrangements of α with no peaks is $(n-1)2^{n-2}$.

n	Non-decreasing PF_n	Rearrangements w/o Peaks	Total Number w/o Peaks
3	$(1,1,2)$	2	
	$(1,1,3)$	2	4
4	$(1,1,2,3)$	4	
	$(1,1,2,4)$	4	
	$(1,1,3,4)$	4	12
5	$(1,1,2,3,4)$	8	
	$(1,1,2,3,5)$	8	
	$(1,1,2,4,5)$	8	
	$(1,1,3,4,5)$	8	32
6	$(1,1,2,3,4,5)$	16	
	$(1,1,2,3,5,6)$	16	
	$(1,1,2,3,4,6)$	16	
	$(1,1,3,4,5,6)$	16	
	$(1,1,2,4,5,6)$	16	80

TABLE 7. The number of rearrangements of $(1, 1, a_3, a_4, \ldots, a_n)$ with no peaks: This table shows the number of non-decreasing $\alpha \in PF_n$ rearrangements that have no peaks, following the format of having two ones and $n-1$ distinct entries.

Proof. If we take the non-decreasing parking functions with $n-1$ distinct entries and two adjacent ones, we can obtain all the rearranged parking functions of length n with of the same conditions that yield. Since there are $n-1$ numbers in $\{2,\dots,n\}$ that can follow the leading ones and $n-2$ entries to place them in, there are

$$\binom{n-1}{n-2} = n-1$$

ways of choosing the $n-2$ distinct numbers to follow the adjacent ones.

Given such a non-decreasing parking function $\alpha = (1,1,a_3,\dots,a_n)$ with exactly two ones and $n-1$ distinct numbers, we now count the number of rearrangements of α with no peaks. The pair of ones must be adjacent by Lemma 5.2. Same as in the proof of Theorem 2.15, the numbers to the left of the adjacent ones must be in decreasing order and the numbers to the right of the pair must be in increasing order so that there are 2^{n-2} rearrangements of α. Thus, there are $(2^{n-2})(n-1)$ parking functions of length n with $n-1$ distinct entries and exactly two repeated ones. □

Example 5.6. Consider the non-decreasing rearrangements of parking functions in PF_4 from Table 6. The parking functions with two ones and $n-1 = 4-1 = 3$ distinct numbers are

$$(1,1,2,4),\ (1,1,2,3) \text{ and } (1,1,3,4).$$

We already know $(1,1,2,3)$ yields four rearrangements with no peaks (refer to Theorem 5.4). However, $(1,1,2,4)$ and $(1,1,3,4)$ also each yield four rearrangements with no peaks. Thus, we have

$$(n-1)2^{n-2} = (4-1)2^{4-2} = 3(2^2) = 3(4) = 12.$$

Hence, the total number of rearrangements with no peaks that have two 1's and $n-1$ distinct numbers is 12. Next, we generalize Theorem 5.4 to extend to parking functions with k ones where the remaining numbers are consecutive and distinct.

Theorem 5.7. Let $\alpha = (\underbrace{1,\dots,1}_{k},2,3,\dots,n)$ be the non-decreasing parking function of length $N = n+k-1$ consisting of k ones followed by the numbers $2,3,\dots,n$. Then there are 2^{N-k} rearrangements of α with no peaks.

Proof. Let $\alpha = (\underbrace{1,\dots,1}_{k},2,3,\dots,n)$ and consider the general shape a rearrangement of α must have to guarantee we have no peaks. Per Lemma 5.2, we must keep all number ones together. Thus, any rearrangement α' can be written

$$\alpha' = \alpha_1\,\underbrace{1,\dots,1}_{k}\,\alpha_2$$

where α_1 denotes the portion to the left of the k number ones and α_2 denotes the portion to the right of the number ones (refer to Figure 6). There will be no peaks if and only if α_1 is decreasing and α_2 is increasing. Let i denote the number of spaces to the left of the first number 1, i.e. the length of α_1. We have i spots to fill and $n-1$ numbers to choose from. Hence, we have $\binom{n-1}{i}$ ways of filling up α_1 in decreasing order. There is only one way to place the remaining $n-i-1$ numbers in increasing order for α_2. If the leftmost number 1 is at index 1, then $i=0$. If the rightmost number 1 is at index $n-k-1$, then $i = n-1$. Therefore, the number of rearrangements of α is

$$\sum_{i=0}^{n-1}\binom{n-1}{i} = 2^{n-1}.$$

Substituting in $n = N+1-k$, the result 2^{N-k} follows. □

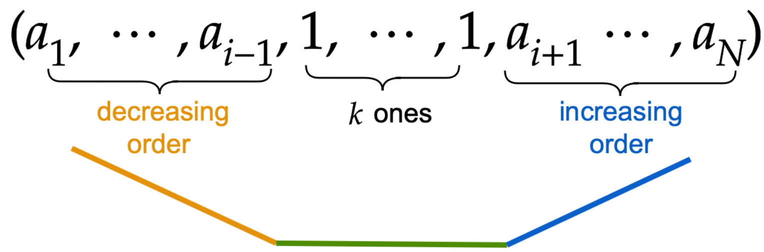


Figure 6. A shape to demonstrate the structure of the special case of k repeating ones.

6. Dyck Paths

In this section, we further analyze non-decreasing parking functions of length n with repeating digits and their rearrangements by studying Dyck paths. This application allows for further observation of patterns based upon which labeled Dyck paths correspond to parking functions with or without peaks.

Definition 6.1. For a given $n \in \mathbb{N}$, a **Dyck Path** of length $2n$ is a lattice path from $(0,0)$ to (n,n) consisting of n steps by $(1,0)$ east and n steps by $(0,1)$ north such that the path never goes above the line $y = x$.

There is a bijection between the number of non-decreasing parking functions of length n and the number of Dyck paths of length $2n$. If $\alpha = (a_1, \ldots, a_n) \in PF_n$, then the corresponding lattice path has east steps from $(i-1, a_i - 1)$ to $(i, a_i - 1)$ at height $a_i - 1$, for each i. The corresponding lattice path for each parking function α does not cross the line $y = x$.

Example 6.2. Consider the non-decreasing parking function $\alpha = (1, 2, 3, 3)$. There is one east step at height 0, one east step at height 1, and two east steps at height 2. Thus, α has the following Dyck path which stays below the dotted line $y = x$ (Figure7):

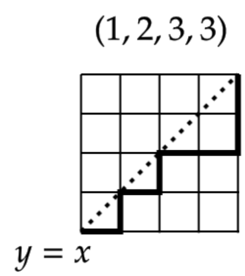
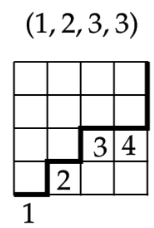


Figure 7. The Dyck path for the parking function $(1, 2, 3, 3)$.

Definition 6.3. A **labeling of a Dyck Path** of length $2n$ is a labeling of the n east steps by a distinct number of $[n]$ such that consecutive east steps are in increasing order.

Note distinct numbers are numbers that are different, though not necessarily unique. Unique numbers are numbers that only occur once in the parking function. For instance, in the parking function $(1, 2, 3, 3)$, 3 is the greatest distinct number, however it occurs twice so it is not unique. In the parking function, $(1, 2, 3, 3, 4)$, 4 is the greatest unique number since it is the highest value that occurs once.



(4 3 12) → (3 3 21)
(3 1 24) → (2 3 13)
(2 4 13) → (3 1 32)
(1 4 23) → (1 3 32)

FIGURE 8. As a continuation of Example 6.2, this figure shows the different labelings of the Dyck path corresponding to $(1, 2, 3, 3) \in PF_4$. The labelings in blue correspond to parking functions with peaks (shown in red).

The Dyck path labelings of a parking function α correspond to rearrangements of α. Since $\alpha \in PF_n$, each rearrangement of α is also a parking function [4]. We now examine the labelings of Dyck paths and their corresponding rearrangements.

Four of the twelve possible rearrangements of the parking function $(1, 2, 3, 3)$ are listed in Figure 8. Notice the labelings have a space in between the first and second index, and a space in between the second and third index. This is to convey that the first two labels are not on consecutive east steps and do not have to be in increasing order. This differs from the third and fourth labels because they are on consecutive east steps.

The labeling (3124) yields the parking function (2313). The numbers in a labeling correspond to each car while their indices correspond to their preferred parking spot on the Dyck path. Hence, car one wants spot 2, car two wants spot 3, car three wants spot 1, and car four wants spot 3. This yields the rearrangement (2313). From this point forward, blue labelings denote those that yield parking functions with a peak. The corresponding peaks in the rearranged parking functions are colored red.

We begin analyzing the bijection between rearranged non-decreasing parking functions and their Dyck path labelings by looking at permutations. We do this to better understand which Dyck paths labelings of permutations correspond to permutations with no peaks. We focus on permutations first since we know the enumeration of those with no peaks. This helps us to generalize patterns for other parking functions. Now, we look at the labeled Dyck paths for permutations in S_3 (Figures 9 and 10):

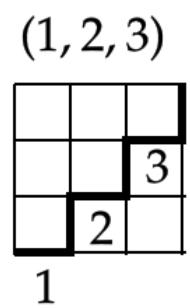


FIGURE 9. The Dyck path for the permutation (123).

Notice the blue labelings have car two in the 3^{rd} index – this yields a parking function with a peak (Figure 10). Car two in the 3^{rd} index means car two prefers the 3^{rd} parking spot. Since the original permutation is in non-decreasing order, the preference of the 3^{rd} index is the greatest unique number. When car two prefers the 3^{rd} spot it places the greatest unique number in the second index. Recall the proof of Lemma 2.11 which justifies that a peak can only occur in indices

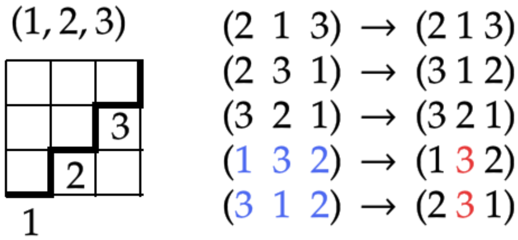


FIGURE 10. The labelings of S_3.

2 through $n-1$. Also, recall the proof of Lemma 2.14 which justifies that if n occurs in the indices 2 through $n-1$ a peak occur. Thus, placing the greatest unique number (3) in the second index creates a peak.

When the second car of any parking function is in the n^{th} index of the parking function's corresponding Dyck path labeling, then a peak occurs. This is the first notable pattern. When there exists an $i \in \{2, \ldots, n-1\}$ where a_i is the greatest unique number, then a peak occurs at i because $a_{i-1} < a_i > a_{i+1}$. Hence, when the second car places the greatest unique number in the second index, a peak occurs. Next we discuss the Dyck paths of parking functions of length 4 with repeating digits by focusing on the repetitions of the number 1.

Consider the parking function $(1,1,1,1) \in PF_4$ where each element equals 1 (Figure 11). This parking function has no distinct rearrangements. We can generalize this parking function to $(1, \ldots, 1)$ where there are n repeated ones. Every step in this parking function is a tie, thus it has no peaks.

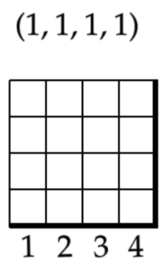


FIGURE 11. The Dyck path for the parking function $(1,1,1,1)$.

Now consider the parking function $(1,1,1,2)$ with three repeating ones and four rearrangements (Figure 12).

This parking function has three repeating ones and a unique number (2). Notice, the blue labelings with car two or car three in the 4^{th} index yield a parking function with a peak. When car two or car three prefer the 4^{th} spot, the unique number (2) is placed at some index in $\{2, \ldots, n-1\}$. This follows the same pattern described in the permutation S_3, where if the second car prefers the n^{th} spot it places the greatest unique number at some index $i \in \{2, \ldots, n-1\}$, creating a peak.

Also, notice the same blue labelings have nonconsecutive numbers in the consecutive east steps of the labeled Dyck path. The non-sequential cars one, three, and four prefer the same spot. This is the same for cars one, two, and four in the other blue labeling. Thus, in the corresponding rearranged parking function, the repeating ones are separated. Since the second or third car prefers the 4^{th} spot, they place the greatest unique number in between the separated ones, creating a peak.

A notable pattern we see is that when there are nonconsecutive numbers in the consecutive east steps of the labeled Dyck path, a peak occurs. This is because cars in non-sequential order prefer

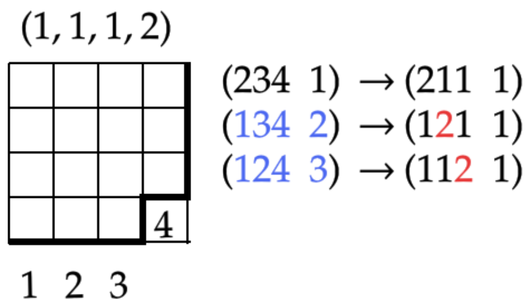


FIGURE 12. The Dyck path and labelings of $(1, 1, 1, 2)$.

the same spot, separating the repetitions in the parking function. Also, when cars in indices 2 through $n-1$ prefer the n^{th} spot, a peak occurs. This is because the cars will place the greatest unique number in the indices where a peak can occur.

We can generalize this parking function for any length n with $k = n-1$ repeating ones and a distinct unique number j, where $j \in \mathbb{N}$ and $j > 1$. A peak not occur when in parking functions are of the form, $(1, \ldots, 1, j), (j, 1, \ldots, 1)$. Notice j is in the first or last index, where a peak cannot occur. Also, a peak not occur for any length n with k repeating ones and $j = n-2$ distinct unique numbers when written in the form described in Theorem 5.7.

Now consider the parking function $(1, 1, 3, 3)$ with two repeating ones and six rearrangements (Figure 13):

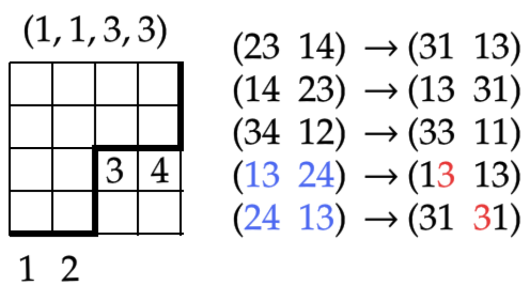


FIGURE 13. The Dyck path and labelings of (1,1,3,3).

This parking function has two repeating ones and two repeating threes. In other words, we have two distinct numbers that are not unique. Notice the blue labelings have nonconsecutive numbers in both consecutive east steps of the labeled Dyck path yielding parking functions with a peak. Every other car prefers the same parking spot. Thus, in the corresponding rearranged parking function, the repetitions are separated. Since the repetitions are separated by a singular index, this causes the repetitions to alternate every other index. These alternations create a peak in the rearranged parking function.

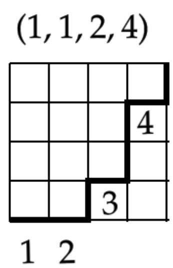
Notice, the labeling (1423) yields the rearranged parking function with no peaks (1331). This labeling has car three in the 4^{th} index and has nonconsecutive numbers in the first set of the consecutive east steps. These conditions created a peak in the parking function $(1, 1, 1, 2)$. However, this is not the case for the parking function $(1, 1, 3, 3)$. The nonconsecutive numbers in the first consecutive east steps separate the repeated ones. Car three places the the greatest distinct number

(3) in the indices $\{2, \ldots, n-1\}$, but the distinct number 3 is not unique. Instead there are two 3's which create a tie in the indices $\{2, \ldots, n-1\}$. Thus, a peak does not occur.

There is a pattern for parking functions with repeating digits of similar form. When there are nonconsecutive numbers in both consecutive east steps of the labeled Dyck path, a peak occurs. This is because every other car prefers the same parking spot, separating the repetitions by a singular index and causing the repetitions to alternate every other index. These alternations create a peak in the rearranged parking function.

A generalization for any even length n would have two distinct numbers where each number repeats $\frac{n}{2}$ times. Notice in order to be a parking function one of the distinct numbers must be 1. A peak occur when the labelings of the Dyck path are separated into odd consecutive numbers and even consecutive numbers for either consecutive east steps of the labeled Dyck path.

Now, consider the parking function $(1, 1, 2, 4)$ with two repeating ones and twelve rearrangements (Figure 14):



(23 1 4) → (21 1 4)
(34 2 1) → (42 1 1)
(23 4 1) → (41 1 2)
(14 3 2) → (14 2 1)
(13 2 4) → (12 1 4)
(24 1 3) → (21 4 1)
(14 2 3) → (12 4 1)
(12 4 3) → (11 4 2)
(34 1 2) → (24 1 1)
(13 4 2) → (14 1 2)
(24 3 1) → (41 2 1)

FIGURE 14. Dyck path for the parking function $(1, 1, 2, 4)$ with the corresponding labelings and rearranged parking fucntions.

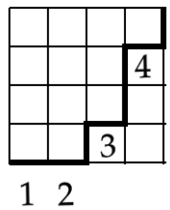
This parking function has three distinct numbers 1, 2, and 4. There are two ones, but both 2 and 4 are unique. There are multiple blue labelings that yield parking functions with a peak. Notice, the blue labelings with car two or car three in the 4^{th} index yield a parking function with a peak. When car two or car three prefer the 4^{th} spot, the greatest unique number (4) is placed in the indices $\{2, \ldots, n-1\}$ creating a peak.

Also notice, blue labelings with nonconsecutive numbers in the first consecutive east steps of the Dyck path yield a parking function with a peak. Cars in non-sequential order preferring the same spot separates the repeating ones. Since the second or third car prefers the 4^{th} spot, they place the greatest unique value in between the separated ones, creating a peak.

Notice the blue labeling and corresponding rearranged parking function, $(1243) \to (1142)$, does not separate the repeating ones, yet a peak occurs (Figure 15). This is due to car three being in the 4^{th} index of the labeling , thus placing the greatest unique number in the indices $i \in \{2, \ldots, n-1\}$ where a peak can occur. Hence, when repeated ones are together, a peak may still occur.

The parking function $(1, 1, 2, 4)$ follows the first pattern of parking functions with repeating digits. When there are nonconsecutive numbers in the consecutive east steps of the labeled Dyck path, a peak occurs. This is because cars in non-sequential order prefer the same spot, separating the repetition in the parking function. Also, when cars in the indices 2 through $n-1$ prefer the

(1,1,2,4)



(23 1 4) → (21 1 4)
(34 2 1) → (42 1 1)
(23 4 1) → (41 1 2)
(14 3 2) → (14 2 1)
(13 2 4) → (12 1 4)
(24 1 3) → (21 4 1)
(14 2 3) → (12 4 1)
(12 4 3) → (11 4 2)
(34 1 2) → (24 1 1)
(13 4 2) → (14 1 2)
(24 3 1) → (41 2 1)

FIGURE 15. The labeled Dyck path, labelings, and rearranged parking functions of (1,1,2,4) focusing on the lableing (12 4 3) and the rearranged parking function (11 4 2)

n^{th} spot, a peak occurs. This is because the cars place the greatest unique number in the indices where a peak can occur.

We can form a generalization of this parking function for any length n with the conditions of $j = n - 2$ unique numbers and two repeating ones. A peak not occur when a parking function is written in the form described in Theorem 5.4.

7. Open Problems

Our original research problem was to find the number of parking functions with no peaks – this is still an open problem. Although we have not yet been able to enumerate this, we have included various observations and properties about peaks in parking functions throughout this paper.

Another interesting research question pertains to Naples parking functions. Naples parking functions are a generalization of parking functions with some new rules, first introduced by Alyson Baumgardner [1]. Car c_i goes to its preferred parking spot a_i. If the spot is taken, it check spot $a_i - 1$. If spot $a_i - 1$ is empty, c_i will park there. Otherwise, c_i continue east (down the one way street) until it reaches an empty parking spot. If all cars can park using the above rules, we call it a Naples parking function. Both the enumeration of Naples parking functions in general and the number of Naples parking functions with no peaks are open problems.

Some other open problems worth investigating are listed below.

(1) How many parking functions of length n have exactly k peaks?
(2) How many parking functions of length n have no valleys?
(3) Given the conditions that a parking function of even length n has two distinct numbers where each number repeats $\frac{n}{2}$ times, a peak occur if there are non consecutive numbers for either consecutive east steps of the labeled Dyck path?
(4) What is the closed formula for the number of parking functions of any length n, if two cars can park in the same spot?

8. **Acknowledgments**

This research was conducted during the 2019 Mathematical Sciences Research Institute Undergraduate Program in Berkeley, CA. This program is supported by the Alfred P. Sloan Foundation, Mathematical Sciences Research Institute, and the National Science Foundation. We would like to thank Prof. Rebecca Garcia (Sam Houston State University) for her mentorship.

References

[1] Alyson Baumgardner, *The naples parking function*, Honors Contract-Graph Theory, Williams College, 2019. ↑

[2] Sara Billey, Krzysztof Burdzy, and Bruce E. Sagan, *Permutations with given peak set*, J. Integer Seq. **16** (2013), no. 6, Article 13.6.1, 18. MR3083179 ↑

[3] P. E. Harris, *Combinatorial problems related to kostant's weight multiplicity formula*, Ph.D. Dissertation, University of Wisconsin Milwaukee (2012). ↑

[4] Alan G Konheim and Benjamin Weiss, *An occupancy discipline and applications*, SIAM Journal on Applied Mathematics **14** (1966), no. 6, 1266–1274. ↑

Appendix A.

In this section we place further information about various observations and properties of peaks in parking functions. We give brief descriptions of each and reference if they pertain to a particular section.

A.1. **Parking functions with no peaks in comparison to parking functions with any peaks.** Table 8 compares parking functions with various quantities. This was an early approach we took to find patterns in the number of parking functions with no peaks by analyzing of our various charts from different angles.

Table 8 shows that the absolute value of the difference between $\frac{\#PF_n}{2}$ and $\#P_*(\emptyset; n)$ is equal to the absolute value of the difference between $\frac{\#PF_n}{2}$ and $\#(PF_n \setminus P_*(\emptyset; n))$. In the table, $K = \frac{\#PF_n}{2}$ and $\#(PF_n \setminus P_*(\emptyset; n))$ is the cardinality of the set of parking functions with *any* peaks.

n	$\#PF_n$	$\frac{\#PF_n}{2} = K$	$\#P_*(\emptyset; n)$	$\#(PF_n \setminus P_*(\emptyset; n))$	$K - \#P_*(\emptyset; n)$	$K - \#(PF_n \setminus P_*(\emptyset; n))$
3	16	8	12	4	4	4
4	125	62.5	59	66	3.5	3.5
5	1,296	648	351	945	297	297
6	16,807	8,403.5	2,499	14,308	5,904.5	5,904.5
7	262,144	131,072	20,823	241,321	110,249	110,249
8	4,782,969	2,391,484.5	197,565	4,585,404	2,193,919.5	2,193,919.5

Table 8. Sequences of K, $\#P_*(\emptyset; n)$ and $\#(PF_n \setminus P_*(\emptyset; n))$

Notice the last two columns of Table 8 are equivalent for all given parking functions of lengths 3 through 8. Further stated, K is equidistant to $A_\emptyset$ and A_c, written as $A_\emptyset \leftarrow K \rightarrow A_c$, where $A_\emptyset = \#P_*(\emptyset; n)$ and $A_c = \#(PF_n \setminus P_*(\emptyset; n))$. So,

$$\begin{aligned} K - A_\emptyset &= A_c - K \\ 2K &= A_\emptyset + A_c \\ (n+1)^{(n-1)} &= A_\emptyset + A_c \\ (n+1)^{(n-1)} &= \#P_*(\emptyset; n) + \#(PF_n \setminus P_*(\emptyset; n)). \end{aligned}$$

Therefore, the number of parking functions is equal to parking functions with no peaks added to parking functions with a peak at any index.

A.2. **Sequences plugged into OEIS.** Here we include a chart of the different sequences we have looked at and searched up on OEIS. These sequences derive from codes we uploaded on GitHub. The following listings describe what set each sequence of elements within the Table 9 belong to.

- $\#P_*(\emptyset; n) \rightarrow$ The number of $\alpha \in PF_n$ with no peaks
- $\#P_R(\emptyset; n) \rightarrow$ The number of $\alpha \in RPF_n$ with no peaks
- $\lceil K - \#P_*(\emptyset; n) \rceil \rightarrow$ The difference of the number of $\alpha \in PF_n$ divided by 2 and the number of $\alpha \in PF_n$ with no peaks
- $\#P_R(\{2\}; n+1) - \#P_R(\{2\}; n) \rightarrow$ The difference between the number of $\alpha \in RPF_n$ with peaks at $i = 2$ of length n and length $n + 1$
- $\#P_R(\emptyset; n)| - n! \rightarrow$ The difference between the number of $\alpha \in RPF_n$ with no peaks and the number of permutations

The Sequence	Elements $n \geq 3$	OEIS
$\#P_*(\emptyset; n)$	12, 59, 351, 2499, 20823	NO RESULTS
$\#P_R(\emptyset; n)$	8, 51, 335, 2467, 20759	NO RESULTS
$\#PF_n - \#P_*(\emptyset; n)$	4, 66, 945, 15862	NO RESULTS
$\lceil K - \#P_*(\emptyset; n) \rceil$	4, 4, 297, 5905, 11024	NO RESULTS
$\lfloor K - \#P_*(\emptyset; n) \rfloor$	4, 3, 297, 5904, 11024	NO RESULTS
$\#P_R(\{2\}; n+1) - \#P_R(\{2\}; n)$	23, 186, 1486, 13137	NO RESULTS
$\#P_R(\emptyset; n) - n^{(n-1)}$	1, 13, 290, 5309	NO RESULTS
$\#P_R(\emptyset; n) - n!$	2, 27, 215, 1747	NO RESULTS

TABLE 9. Chart of Sequences

A.3. **Maximum Peaks in Parking functions.** The following chart explores a pattern of the maximum number of peaks that can occur in a parking function of a length n. Specifically, Table 10 focuses on odd length n from length $n = 1$ to length $n = 19$. This was an approach in our research where we decided to analyze where peaks can occur and how many occur in parking functions.

n is odd	Max Peaks	Closed Form = ???	Pattern
1	0	$n-1$	$n-1$
3	1	$n-2$	$(n-1)-1$
5	2	$n-3$	$(n-1)-1-1$
7	3	$n-4$	$(n-1)-1-1-1$
9	4	$n-5$	$(n-1)-1-1-1-1$
11	5	$n-6$	$(n-1)-1-1-1-1-1$
13	6	$n-7$	$(n-1)-1-1-1-1-1-1$
15	7	$n-8$	$(n-1)-1-1-1-1-1-1-1$
17	8	$n-9$	$(n-1)-1-1-1-1-1-1-1-1$
19	9	$n-10$	$(n-1)-1-1-1-1-1-1-1-1-1$

TABLE 10. Max number of peaks for a parking function (length n, n odd)

A.4. **Trends of non-decreasing parking functions with repeating digits.** Table 11 displays more trends of non-decreasing parking functions with repeating digits. Particularly, parking functions with repeating digits of length $n = 5$. This is work expands on our examination of the behavior of repeating digits in Section 5.

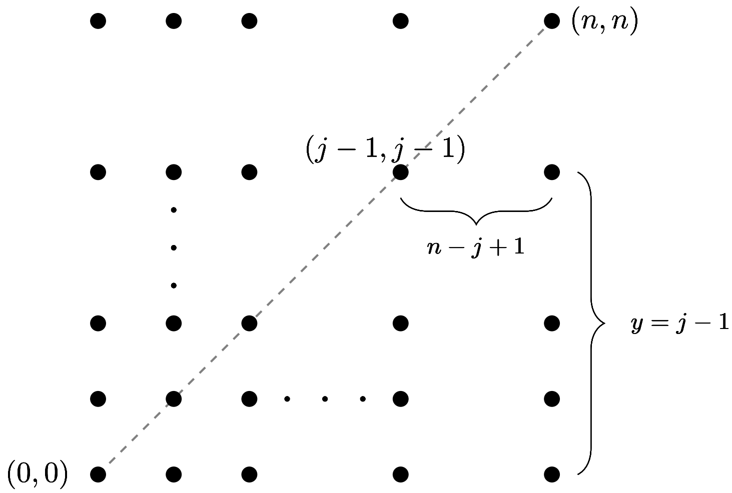
Non-decreasing $\alpha \in PF_5$	w/o Peaks	w/ Peaks	Trends
(1, 1, 1, 1, 1)	1		One digit repeated 5x
(1, 1, 1, 1, 2)	2		One digit repeated 4x
(1, 1, 1, 1, 3)	2		One digit repeated 4x
(1, 1, 1, 1, 4)	2		One digit repeated 4x
(1, 1, 1, 1, 5)	2		One digit repeated 4x
(1, 1, 1, 2, 3)	4		One digit repeated 3x
(1, 1, 1, 2, 4)	4		One digit repeated 3x
(1, 1, 1, 2, 5)	4		One digit repeated 3x
(1, 1, 1, 3, 4)	4		One digit repeated 3x
(1, 1, 1, 3, 5)	4		One digit repeated 3x
(1, 1, 1, 4, 5)	4		One digit repeated 3x
(1, 2, 2, 2, 2)	5		One digit repeated 4x
(1, 1, 1, 2, 2)	5		One digit repeated 3x and One digit repeated 2x
(1, 1, 1, 3, 3)	5		One digit repeated 3x and One digit repeated 2x
(1, 1, 1, 4, 4)	5		One digit repeated 3x and One digit repeated 2x
(1, 1, 2, 2, 2)	7		One digit repeated 3x and One digit repeated 2x
(1, 1, 3, 3, 3)	7		One digit repeated 3x and One digit repeated 2x
(1, 1, 2, 2, 3)	8		Two digits repeated 2x
(1, 2, 2, 2, 4)	8		One digit repeated 3x
(1, 2, 2, 2, 3)	8		One digit repeated 3x
(1, 1, 2, 2, 4)	8		Two digits repeated 2x
(1, 2, 2, 2, 5)	8		One digit repeated 3x
(1, 1, 2, 3, 4)	8		One digit repeated 2x
(1, 1, 2, 3, 5)	8		One digit repeated 2x
(1, 1, 2, 4, 5)	8		One digit repeated 2x
(1, 1, 3, 3, 4)	8		Two digits repeated 2x
(1, 1, 3, 3, 5)	8		Two digits repeated 2x
(1, 1, 3, 4, 5)	8		One digit repeated 2x
(1, 1, 2, 3, 3)	12		Two digits repeated 2x
(1, 1, 2, 4, 4)	12		Two digits repeated 2x
(1, 2, 2, 3, 4)	12		One digit repeated 2x
(1, 2, 2, 3, 5)	12		One digit repeated 2x
(1, 2, 2, 4, 5)	12		One digit repeated 2x
(1, 1, 3, 4, 4)	12		Two digits repeated 2x
(1, 2, 3, 3, 3)	14		One digit repeated 3x
(1, 2, 2, 3, 3)	15		Two digits repeated 2x
(1, 2, 2, 4, 4])	15		Two digits repeated 2x
(1, 2, 3, 3, 4)	16		One digit repeated 2x
(1, 2, 3, 3, 5)	16		One digit repeated 2x
(1, 2, 3, 4, 4)	24		One digit repeated 2x

TABLE 11. Non-decreasing parking functions of length $n = 5$ and the trends of repeating digits

A.5. **The number of times a digit can repeat based on a length n.** The following is a proof of how many times a number can be repeated in a parking function of length n based upon the definition of a Dyck path. A parking function where every element is a 1, $(1, 1, 1, \ldots, 1)$, has the maximum number of a repeating digits. This proof determines how many times other numbers can repeat to still be a parking function by staying below the line $y = x$ in the Dyck path.

Lemma A.1. Consider a parking function $\alpha \in PF_n$, then the max number of entries equal to some $j \in [n]$ is $n - j + 1$.

Proof. Let $\alpha \in PF_n,\ \alpha = (a_1 \le a_2 \le \cdots \le a_n)$. There is a bijection between the set of increasing parking functions of length n and Dyck paths.



Since we assumed $\alpha \in PF_n$ the corresponding Dyck path must remain below the line $y = x$. Consequently at any height j, y-value $j - 1$, there can be at most $n - (j - 1) = n - j + 1$ east steps from $(j - 1, j - 1)$ to $(n, j - 1)$. ☐

California State University Channel Islands, Department of Mathematics, United States
E-mail address: lovatic27@gmail.com

Spelman College, Department of Mathematics, United States
E-mail address: summartime08@gmail.com

Department of Mathematics and Statistics, Williams College, United States
E-mail address: peh2@williams.edu

Spelman College, Department of Mathematics, United States
E-mail address: morgan.hobson22@gmail.com

Department of Mathematics, University of Illinois at Urbana-Champaign, United States
E-mail address: mloving2@illinois.edu

Department of Mathematics, University of Illinois at Urbana-Champaign, United States
E-mail address: rennie2@illinois.edu

UC Santa Barbara, Department of Mathematics, United States
E-mail address: gkirby@math.ucsb.edu